

Minkowski momentum of an MHD wave

Tadas K Nakamura

CFAAS, Fukui Prefectural University, 910-1195 Fukui, Japan

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Abstract. The momentum of an MHD wave has been examined from the view point of the electromagnetic momentum expression derived by Minkowski. Basic calculations show that the Minkowski momentum is the sum of electromagnetic momentum and the momentum of the medium, as proposed in some of the past literature. The result has been explicitly confirmed by an example of an MHD wave, whose dynamics can be easily and precisely calculated from basic equations. The example of MHD wave also demonstrates the possibility to construct a symmetric energy-momentum tensor based on the Minkowski momentum.

1. Introduction

The Minkowski-Abraham controversy has been discussed by a number of authors over a hundred years. Minkowski [1] proposed the electromagnetic momentum density in a dielectric medium must be $\mathbf{D} \times \mathbf{B}$, and Abraham [2, 3] proposed $\mathbf{E} \times \mathbf{H}$ for that (in the present letter symbols have conventional meanings, e.g., \mathbf{E} = electric field, unless otherwise stated). There have been published numerous papers on this problem both theoretically and experimentally, but the final conclusion is still yet to come; papers are still being published in this century (see, e.g., [4] for a review).

Several authors [5, 4, 6] pointed that the electromagnetic field inevitably affect the dynamics of the medium to change its energy-momentum, and therefore, the energy-momentum of an electromagnetic wave must include the contribution of the medium. In the present letter we show that the Minkowski momentum is the sum of electromagnetic momentum and the momentum of the medium. Feigel [5] obtained a similar result based on the Noether's theorem using Lagrangian formulation. Compared to his elegant approach, the calculation here is rather a down-to-earth type, which is more closer to the Minkowski's original derivation. This approach is less elegant, however, easier to understand its meaning intuitively.

Perhaps the largest weak point of the Minkowski momentum is the fact that the four dimensional energy-momentum tensor does not become symmetric with this momentum, which means the violation of angular momentum conservation (see, e.g. [7]). Most of the past literature argued the legitimacy of the momentum part of the tensor in this point. Here, in contrast, we elucidate the possibility to alter the energy part to make a symmetric tensor; provided the momentum part of the Minkowski energy-momentum tensor includes the momentum of medium, the same should be true for the energy part. To treat it in a relativistically consistent way, the mass flux must be included in the energy flux even in the non-relativistic regime. The energy-momentum tensor with Minkowski momentum can become symmetric when the mass flux is taken into account.

The consideration stated above is confirmed by an example of an MHD wave in a collisionless magnetized plasma. Usually the behavior of an ordinary medium is complicated and need to calculate microscopic states of molecules, which is difficult to solve exactly. A collisionless plasma is, in contrast, easy to calculate its response to the electromagnetic field from the classical basic equations (Maxwell equations and Newtonian mechanics). Here in this short letter we use the MHD approximation, however, if one wishes it is possible to derive an exact solution of the basic equation system to confirm the result. The result agree with the "frozen-in" of a magnetized plasma, which has been confirmed by a wide variety of experimental and observational facts.

2. Basics

Microscopic Ampere's equation in a medium is

$$-\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0^{-1} \nabla \times \mathbf{B} = \mathbf{J} \quad (1)$$

Suppose there is no external current, and \mathbf{J} consists of the polarization current \mathbf{J}_P and magnetization current \mathbf{J}_M , which are generated in response to the electric field \mathbf{E} and magnetic field \mathbf{B} respectively. We introduce the polarization vector \mathbf{P} and magnetization vector \mathbf{M} such that

$$\frac{\partial}{\partial t} \mathbf{P} = \mathbf{J}_P, \quad \nabla \times \mathbf{M} = \mathbf{J}_M \quad (2)$$

In this context, \mathbf{P} and \mathbf{M} should be understood as convenient mathematical expressions to represent the response of the medium to the electromagnetic field, rather than real physical entities.

When averaged over a microscopically large but macroscopically small volume, $\bar{\mathbf{P}}$ and $\bar{\mathbf{M}}$ are assumed to have simple linear relations to the electromagnetic fields as

$$\bar{\mathbf{P}} = \chi_P \bar{\mathbf{E}}, \quad \bar{\mathbf{M}} = \chi_M \bar{\mathbf{B}}. \quad (3)$$

The linear coefficients χ_P and χ_M are matrices in general because the medium may not be isotropic (as in our example of magnetized plasmas). The fields $\bar{\mathbf{D}}$ and $\bar{\mathbf{H}}$ are then defined as macroscopic quantities as

$$\bar{\mathbf{D}} = \varepsilon_0 \bar{\mathbf{E}} + \bar{\mathbf{P}} = \varepsilon \bar{\mathbf{E}}, \quad \bar{\mathbf{H}} = \mu_0^{-1} \bar{\mathbf{B}} + \bar{\mathbf{M}} = \mu^{-1} \bar{\mathbf{B}} \quad (4)$$

3. Minkowski Momentum

The momentum of microscopic electromagnetic field is $\varepsilon_0 \mathbf{E} \times \mathbf{B}$ and its conservation law is

$$\frac{\partial}{\partial t} (\varepsilon_0 \mathbf{E} \times \mathbf{B}) + \nabla \cdot \mathbf{T} + (\mathbf{J}_P + \mathbf{J}_M) \times \mathbf{B} + (Q_P + Q_M) \mathbf{E} = 0, \quad (5)$$

where \mathbf{T} is the Maxwell stress tensor and we denote $\partial T_{ij} / \partial x_i = (\nabla \cdot \mathbf{T})_j$ in short. The polarization/magnetization charge Q_P and Q_M are the result of polarization/magnetization current ($\partial Q_{P,M} / \partial t = \nabla \cdot \mathbf{J}_{P,M}$). The charge due to magnetization current vanishes when averaged, $\bar{Q}_M = 0$ since $\bar{\mathbf{J}}_M$ satisfies (2).

The third and fourth term of (5) are the Lorentz and Coulomb force acting on the medium, and therefore, it can be expressed by the momentum change of the medium.

$$(\mathbf{J}_P + \mathbf{J}_M) \times \mathbf{B} + (Q_P + Q_M) \mathbf{E} = \frac{\partial}{\partial t} \mathbf{g} + \nabla \cdot \mathbf{T}_M, \quad (6)$$

where \mathbf{g} and \mathbf{T}_M are the momentum density and stress tensor of the medium. It should be noted that the right hand side of the above expression has mathematical ambiguity. If we define new values of momentum/stress by $\mathbf{g}' = \mathbf{g} + \mathbf{a}$ and $\mathbf{T}' = \mathbf{T} + \mathbf{G}$ with arbitrary vector \mathbf{a} and tensor \mathbf{G} that satisfy $\partial \mathbf{a} / \partial t = \nabla \cdot \mathbf{G} = 0$, they also satisfy the above equation.

Therefore, \mathbf{g} and T do not necessarily have to be the total momentum/stress of the medium. For example, the medium may contain a part that does not interact with the electromagnetic field, and such part causes this ambiguity.

From (4) we obtain

$$\begin{aligned}\bar{\mathbf{J}}_P \times \bar{\mathbf{B}} &= \frac{\partial}{\partial t}(\bar{\mathbf{P}} \times \bar{\mathbf{B}}) + \chi_P \left[(\bar{\mathbf{E}} \cdot \nabla) \bar{\mathbf{E}} + \frac{1}{2} \nabla \bar{\mathbf{E}}^2 + \bar{\mathbf{E}}(\nabla \bar{\mathbf{E}}) \right], \\ \bar{\mathbf{J}}_M \times \bar{\mathbf{B}} &= \chi_M \left[(\bar{\mathbf{B}} \nabla) \bar{\mathbf{B}} + \frac{1}{2} \nabla \bar{\mathbf{B}}^2 \right].\end{aligned}\quad (7)$$

Here we neglected cross terms of fluctuation in averaging as usually done in this kind of calculation, e.g., $\overline{\mathbf{P} \times \mathbf{B}} = \bar{\mathbf{P}} \times \bar{\mathbf{B}}$. Combining (5), (6) and (7) we obtain

$$\frac{\partial}{\partial t}(\varepsilon_0 \bar{\mathbf{E}} \times \bar{\mathbf{B}} + \bar{\mathbf{g}}) + \nabla \cdot (\bar{\mathbf{T}} + \bar{\mathbf{T}}_M) = \frac{\partial}{\partial t}(\bar{\mathbf{D}} \times \bar{\mathbf{B}}) + \nabla \cdot \bar{\mathbf{T}}' = 0, \quad (8)$$

where $\bar{\mathbf{T}}'$ is the stress tensor in a dielectric medium defined as

$$\bar{T}'_{ij} = \bar{E}_i \bar{D}_j + \mu_0^{-1} \bar{B}_i \bar{H}_j - \frac{1}{2} \delta_{ij} (\bar{\mathbf{E}} \cdot \bar{\mathbf{D}} + \bar{\mathbf{B}} \cdot \bar{\mathbf{H}}), \quad (9)$$

which is the sum of the fluxes of electromagnetic momentum and momentum carried by the medium.

Now that we understand the Minkowski momentum includes the part of the medium then so should be for the energy. The energy is equivalent to mass in relativity, thus the energy flux of the medium must include mass, and then the flux may take the form of $\bar{\mathbf{D}} \times \bar{\mathbf{B}}$ to make the energy momentum symmetric. We will check this for the case of an Alfven wave in the following.

4. MHD wave

Let us confirm the above discussion with an example of an MHD wave. Suppose a linearly polarized one dimensional ($\partial/\partial x = \partial/\partial y = 0$) MHD wave (Alfven wave in this case) propagating in the z direction, which is the direction of the background magnetic field: $B_0 = B_z$. The wave amplitude is small enough for linear approximation, and the plasma velocity is low enough for non-relativistic approximation. Also we assume “cold plasma limit”, which means the thermal energy of plasma particles is negligibly small.

The wave has an electric field perpendicular to its propagation direction, and we take the x axis in this electric field direction. Then the current is also in the x direction, whereas the magnetic perturbation and the plasma velocity is in the y direction (see Appendix). The magnetic field created by cyclotron motion of plasma particles is negligible in a cold plasma limit, and thus we treat the case with $\mu = \mu_0$ hereafter.

The current is the polarization current due to the temporal change of the electric field, which is

$$J_z = \frac{\mu_0}{V_A^2} \frac{\partial}{\partial t} E_x \quad (10)$$

where V_A is the Alfvén speed defined by $V_A = B_0 / \sqrt{\mu_0 \rho}$ with ρ being the mass density of the plasma. From (2) and the above expression we obtain

$$D_x = \varepsilon_0 \left(1 + \frac{c^2}{V_A^2} \right) E_x \quad (11)$$

The plasma frozen-in condition $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$ means that the plasma is moving in the y direction with the $E \times B$ drift speed as

$$v_y = \frac{E_x}{B_0}. \quad (12)$$

Then the momentum carried by the plasma particles is

$$\rho v_y = \frac{\mu_0 B_0}{V_A^2} E_x. \quad (13)$$

The y component of the Minkowski momentum can be calculated from (11) and (13), which is

$$(\mathbf{D} \times \mathbf{B})_y = D_x B_0 = \varepsilon_0 E_x B_0 + \rho v_y. \quad (14)$$

The above expression means the Minkowski momentum is the sum of electromagnetic momentum and momentum of the plasma particles as long as the frozen-in condition is satisfied.

When multiplied by c^2 , the first term of the right hand side of (14) becomes the electromagnetic energy flux in the y direction. The second term becomes the relativistic energy comes from the rest mass, which is the predominant energy flux in the non-relativistic limit here; thermal or kinetic energy flux is negligible. The balance equation of the energy-momentum tensor is in a derivative form, and therefore, it has an ambiguity as discussed below (6) for the momentum. The same is true for the energy, and its conservation also holds when we add the mass density and mass flux $(\rho, \rho \mathbf{v})$, since $\partial \rho / \partial t + \nabla(\rho \mathbf{v}) = 0$. The energy momentum tensor becomes symmetric with these terms.

5. Concluding Remarks

Momentum carried by an electromagnetic wave has been examined with an example of an MHD wave. It has been shown that the total momentum (electromagnetic momentum plus momentum of the medium) is expressed by $\mathbf{D} \times \mathbf{B}$ as proposed by Minkowski. This result is based on the very simple and basic two properties of an MHD plasma, the frozen-in condition (12) and polarization current (10), namely. It would not be exaggeration if one says the whole kingdom of MHD plasma physics would fall if these two basic properties were wrong.

Here in this letter we examined a simplest case of an parallel (to the \mathbf{B} field) propagating MHD wave, but similar calculations can be done for more complicated plasma waves to confirm the result here. A collisionless plasma contains a wide variety of wave phenomena, and the properties of waves can be precisely calculated

at least in the linear limit. Calculation of the Minkowski momentum for various plasma waves would be a good exercise to understand the Abraham-Minkowski controversy.

The drawback of the Minkowski momentum has been believed that the momentum fails to form a symmetric four dimensional energy-momentum tensor when coupled with the Poynting flux; an asymmetric energy-momentum tensor means the violation of angular momentum conservation. This difficulty can be overcome when we include the mass flux as a part of energy flux, which is reasonable from the relativistic point of view. The energy-momentum tensor can be symmetric as we have examined with an MHD wave here.

What we have shown in the present letter is that the Minkowski momentum can be self consistent description of the total momentum of an electromagnetic wave in a polarizable medium. This does not necessarily mean the Abraham momentum is wrong and inconsistent; it might be possible to give Abraham momentum another appropriate meaning to make it consistent. For example, Barnett [8] recently argued both Abraham and Minkowski momentum can be consistent when we interpret the former as kinetic momentum and latter as canonical momentum. It is out of our scope here to examine this argument, however, it should be noted the legitimacy of the Minkowski momentum does not automatically exclude the validity of the Abraham momentum.

Appendix

This appendix is to derive (10) and (12) in a shortest way for a physicist not familiar with plasma physics. For further information, see any textbook on plasma physics, e.g., [9, 10]. Note that many books derive Alfvén waves from the MHD equations, which is different from the derivation here; of course the result is the same.

Suppose a plasma consists of equal number of protons and electrons in a uniform magnetic field, which is in the z direction of Cartesian coordinates. The plasma response to the electromagnetic field can be expressed by \mathbf{P} only and we do not need the magnetization current for our calculation. Therefore we can set $\mathbf{H} = \mu_0^{-1}\mathbf{B}$ here. We assume an MHD wave described above is propagating in this plasma.

Let us denote a vector in the xy plane by a complex number as $A = A_x + iA_y$. Then the wave electric field in the x direction is denoted as

$$E(t) = \frac{E_0}{2}(e^{-i\omega t} + e^{i\omega t}). \quad (15)$$

The equation of motion of a plasma particle in the xy plane is written as

$$\frac{dv}{dt} = i\Omega v + \frac{e}{m}E(t), \quad (16)$$

where $\Omega = eB/m$ is the gyro frequency. We include the sign of the charge in Ω , thus Ω is positive/negative for a proton/electron. The above equation can be directly solved

as

$$v = v_0 e^{i\Omega t} + \frac{eE_0}{2m} \left(\frac{e^{-i\omega t}}{\Omega - \omega} + \frac{e^{i\omega t}}{\Omega + \omega} \right), \quad (17)$$

where v_0 is the integration constant.

Now we assume the wave frequency is much smaller than the gyro frequency ($\omega \ll \Omega$), which is true for most of MHD waves. Then the effect of the first term in (17) will be averaged out for MHD time scale; we do not pay attention to this term hereafter.

The rest of the motion is called “drift” in plasma physics. The drift velocity v_d can be expanded as

$$v_d = \frac{E_0}{B} \left(i \cos \omega t + \frac{\omega}{\Omega} \sin \omega t + \dots \right). \quad (18)$$

The first term of (18) is called $E \times B$ drift; protons and electrons drift in the same direction with the same speed with this drift. This term is pure imaginary, which means the drift is in the y direction. This drift gives the predominant motion of the bulk plasma as in (12), however, it does not cause a current because both protons and electrons have the same drift velocity. What contribute to a current is the second term of (18), which is called the polarization drift. Since this term contains Ω^{-1} factor, protons and electrons moves in the opposite direction with different speed. The electron gyro frequency is much larger than that of protons, therefore, protons predominantly carry currents. The drift direction is the same as the electric field since it is pure real, and the drift speed is proportional to time derivative of the field because of the factor ω and $\cos \omega t \rightarrow \sin \omega t$. Multiplying the proton's second term of (18) with the number density and charge, and replacing the factor ω and $\cos \omega t \rightarrow \sin \omega t$ by the time derivative, we obtain (18).

From the Maxwell's equation we have

$$\nabla \times \nabla \times \mathbf{E} = c^{-2} \partial^2 \mathbf{E} / \partial t^2 + \mu_0 \mathbf{J}. \quad (19)$$

When we assume the wave propagation is in the z direction ($\partial/\partial x = \partial/\partial y = 0$) and use (10), we obtain the propagation equation of an MHD wave (Alfven wave) as

$$\left(\frac{1}{c^2} + \frac{1}{V_A^2} \right) \frac{\partial^2}{\partial t^2} E_x - \frac{\partial^2}{\partial z^2} E_x = 0. \quad (20)$$

The Alfven speed V_A is often much smaller than the speed of light in space and laboratory plasmas. We obtain an wave propagating with the Alfven speed V_A when we ignore the $1/c^2$ term in the above expression.

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